alex-levesque.com

MTHE 281 — Introduction to Real Analysis

Winter 2026

Based on lectures by T. Barthelmé and N. Paulet – Queen's University

Notes written by Alex Lévesque

These notes are my own interpretations of the course material and they are not endorsed by the lecturers.
Feel free to reach out if you point out any errors.

Contents

1 Preface

Grading Scheme:

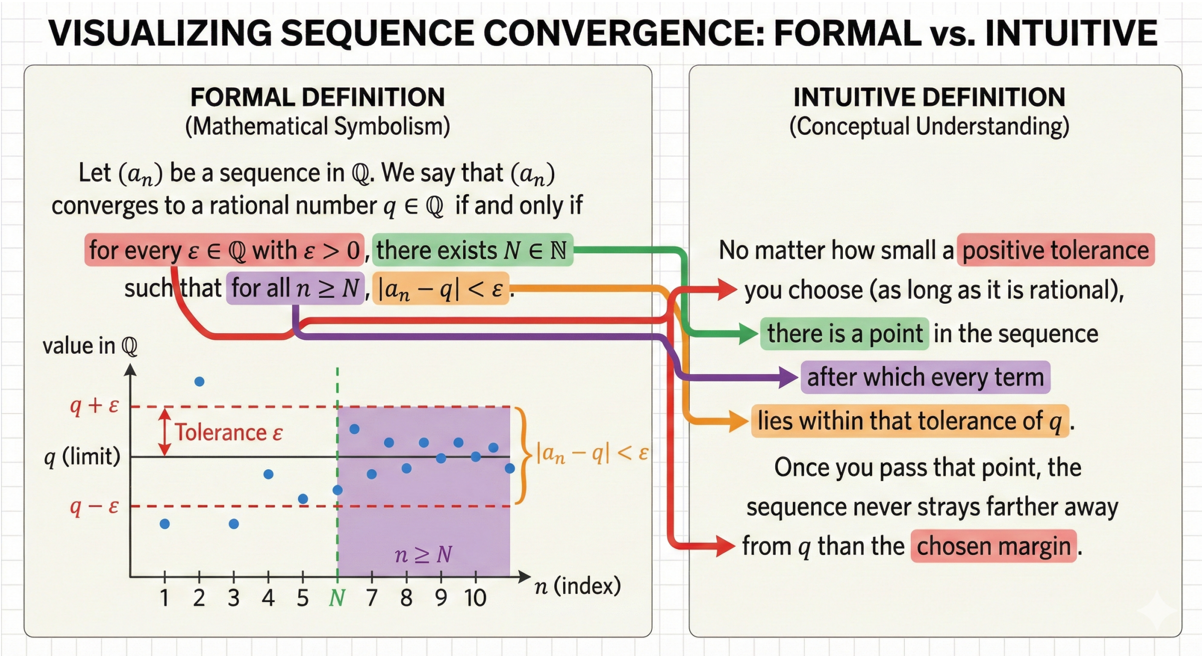
Textbook:

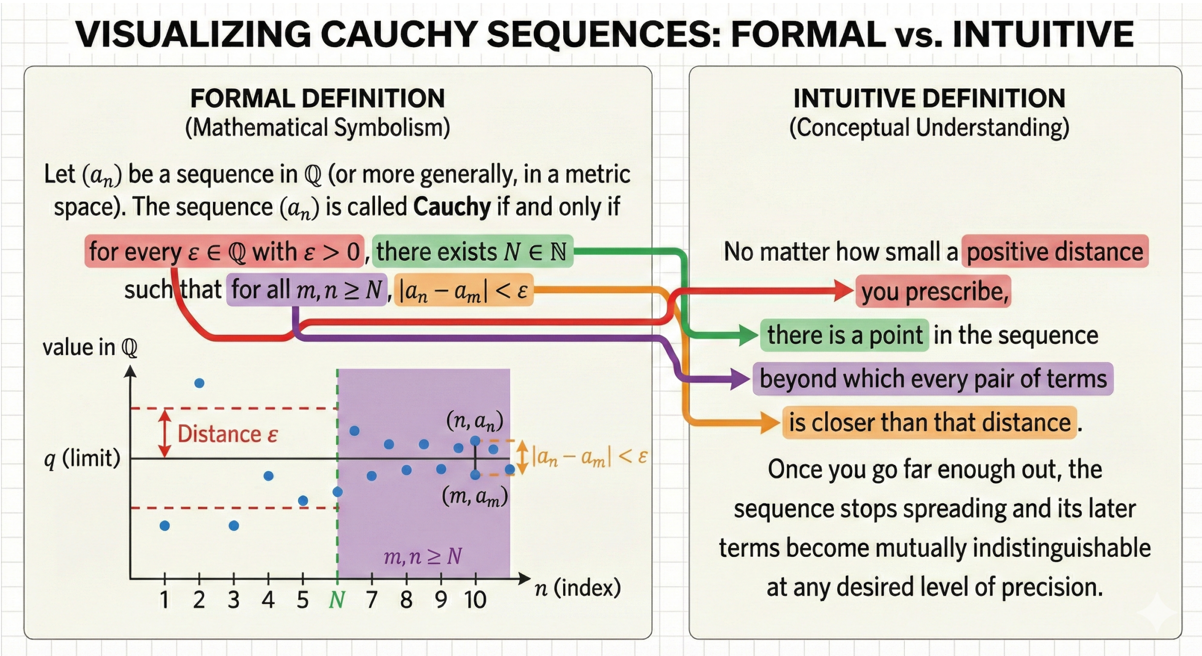
Comments:

Read these: https://www.maths.cam.ac.uk/undergrad/files/studyskills.pdf

https://tll.mit.edu/teaching-resources/how-people-learn/metacognition/

2 Cauchy Sequences





2.1 E1.4: Proof of Boundedness in $\mathbb{Q}$

Proposition: A subset $A \subset \mathbb{Q}$ is bounded if and only if it has a lower bound and an upper bound.

Proof:

$(\Longrightarrow)$

Suppose $A \subset \mathbb{Q}$ is bounded. By definition, there exists some $M > 0$ such that for all $a \in A$, $|a| \leq M$. The inequality $|a| \leq M$ may be rewritten as:

$$-M \leq a \leq M \text{ for all } a \in A$$

This inequality shows that M is an upper bound for A and $-M$ is a lower bound for A.

$(\Longleftarrow)$

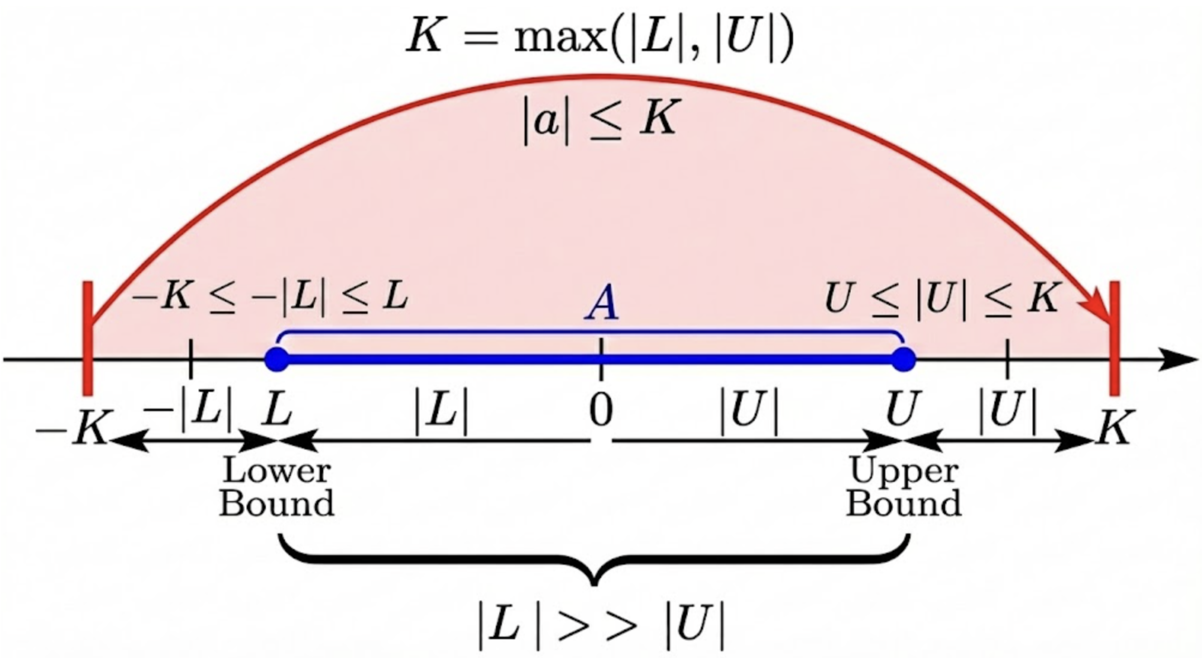
Suppose A has an upper bound U such that $\forall a \in A, a \leq U$, and a lower bound L such that $\forall a \in A, L \leq a$. We may write:

$$\forall a \in A, L \leq a \leq U$$

To show that A is bounded, we must find a single constant $k > 0$ such that $|a| \leq k$ for all $a \in A$.

Let $k = \max(|L|, |U|)$. Then:

- Since $k \geq |L|$, it follows that $-k \leq -|L|$. Because $-|L| \leq L$, we have $-k \leq L$.
- Since $k \geq |U|$ and $U \leq |U|$, we have $U \leq k$.



Therefore, we can write:

$$-k \leq L \leq a \leq U \leq k \Rightarrow -k \leq a \leq k$$

By the properties of absolute value, this implies $|a| \leq k$ for all $a \in A$. Thus, by definition, A is bounded.

■

2.2 E1.6 (b): Epsilon Test for Infimums

Set: $A = \left\{ \frac{\sqrt{x-1}}{x} \mid x \geq 2 \right\}$

1. ***Supremum (*** $\sup A$ ***)***

- **Value:** $\sup A = 1/2$
- **Finding the bound:** The maximum occurs when $x = 2$. Plugging this in: $\frac{\sqrt{2-1}}{2} = \frac{1}{2}$.
- Epsilon Proof: For any given $\varepsilon > 0$, we can choose the number $1/2$ from the set (where $x = 2$). Since ε is positive, it is always true that:

$$1/2 > 1/2 - \varepsilon$$

2. ***Infimum (*** $\inf A$ ***)***

- **Value:** $\inf A = 0$
- **Finding the bound:** The value of the expression approaches 0 as $x \to \infty$.
- Epsilon Proof: For any given $\varepsilon > 0$, we need to find a number in the set such that:

$$\frac{\sqrt{x-1}}{x} < 0 + \varepsilon$$

To simplify the search for x, we note that $\frac{\sqrt{x}}{x} > \frac{\sqrt{x-1}}{x}$. Therefore, any x that satisfies $\frac{\sqrt{x}}{x} < \varepsilon$ will also satisfy our condition.

Simplifying the expression:

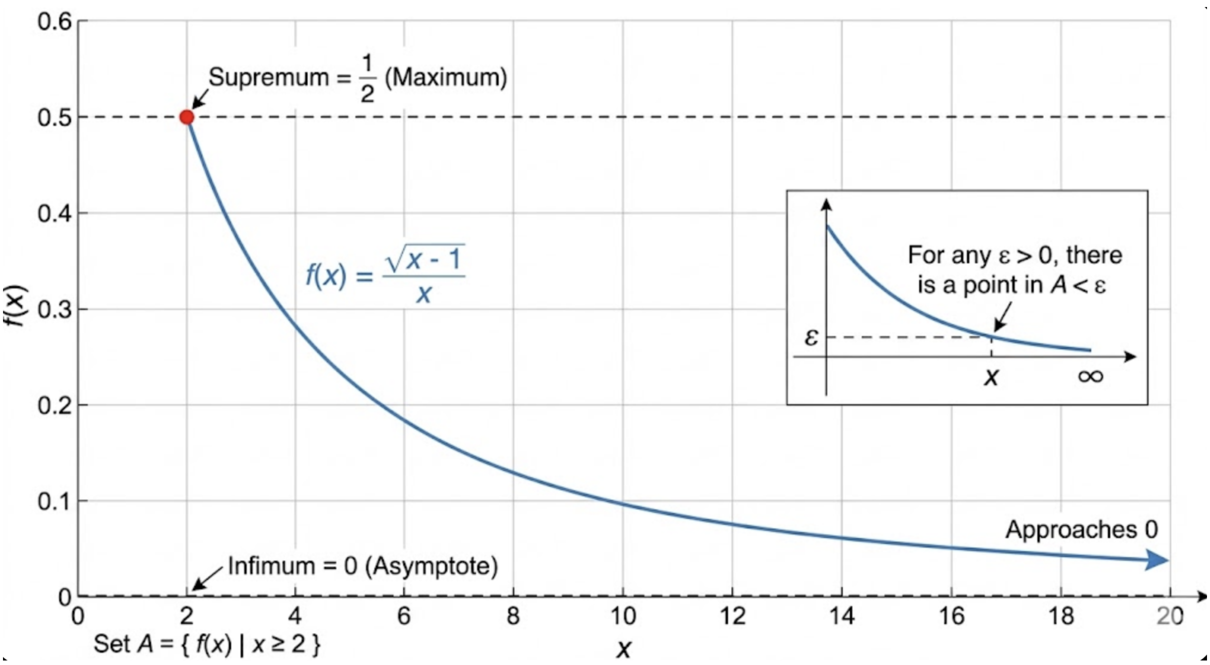
$$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

Solving for x:

$$\frac{1}{\sqrt{x}} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < \sqrt{x} \Rightarrow \frac{1}{\varepsilon^2} < x$$

Choice of x: Pick $x = \frac{1}{\varepsilon^2} + 2$.

This value is valid because it is ≥ 2 (staying in the set) and is strictly $> \frac{1}{\varepsilon^2}$. This ensures the resulting value in the set is smaller than $0 + \varepsilon$.



0.6
0.5
0.4
0.3
0.2
0.1
0
f(x)
Supremum = 1/2 (Maximum)
f(x) = √(x - 1) / x
For any ε > 0, there
is a point in A < ε
ε
x
∞
Approaches 0
Infimum = 0 (Asymptote)
0
2
4
6
8
10
12
14
16
18
20
Set A = { f(x) | x ≥ 2 }
x