

MTHE 281 — Introduction to Real Analysis

WINTER 2026

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These notes are my own interpretations of the course material and they are not endorsed by the lecturers.
Feel free to reach out if you point out any errors.

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1 Preface

Grading Scheme:

Textbook:

Comments:

Read these: <https://www.maths.cam.ac.uk/undergrad/files/studyskills.pdf>

<https://tll.mit.edu/teaching-resources/how-people-learn/metacognition/>

2 Cauchy Sequences

VISUALIZING SEQUENCE CONVERGENCE: FORMAL vs. INTUITIVE

FORMAL DEFINITION
(Mathematical Symbolism)

Let (a_n) be a sequence in \mathbb{Q} . We say that (a_n) converges to a rational number $q \in \mathbb{Q}$ if and only if

for every $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - q| < \varepsilon$.

INTUITIVE DEFINITION
(Conceptual Understanding)

No matter how small a positive tolerance you choose (as long as it is rational), there is a point in the sequence after which every term lies within that tolerance of q .

Once you pass that point, the sequence never strays farther away from q than the chosen margin.

VISUALIZING CAUCHY SEQUENCES: FORMAL vs. INTUITIVE

FORMAL DEFINITION
(Mathematical Symbolism)

Let (a_n) be a sequence in \mathbb{Q} (or more generally, in a metric space). The sequence (a_n) is called **Cauchy** if and only if

for every $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_n - a_m| < \varepsilon$.

INTUITIVE DEFINITION
(Conceptual Understanding)

No matter how small a positive distance you prescribe, there is a point in the sequence beyond which every pair of terms is closer than that distance.

Once you go far enough out, the sequence stops spreading and its later terms become mutually indistinguishable at any desired level of precision.

2.1 E1.4: Proof of Boundedness in \mathbb{Q}

Proposition: A subset $A \subset \mathbb{Q}$ is bounded if and only if it has a lower bound and an upper bound.

Proof:

(\Rightarrow)

Suppose $A \subset \mathbb{Q}$ is bounded. By definition, there exists some $M > 0$ such that for all $a \in A$, $|a| \leq M$. The inequality $|a| \leq M$ may be rewritten as:

$$-M \leq a \leq M \text{ for all } a \in A$$

This inequality shows that M is an upper bound for A and $-M$ is a lower bound for A .

(\Leftarrow)

Suppose A has an upper bound U such that $\forall a \in A, a \leq U$, and a lower bound L such that $\forall a \in A, L \leq a$. We may write:

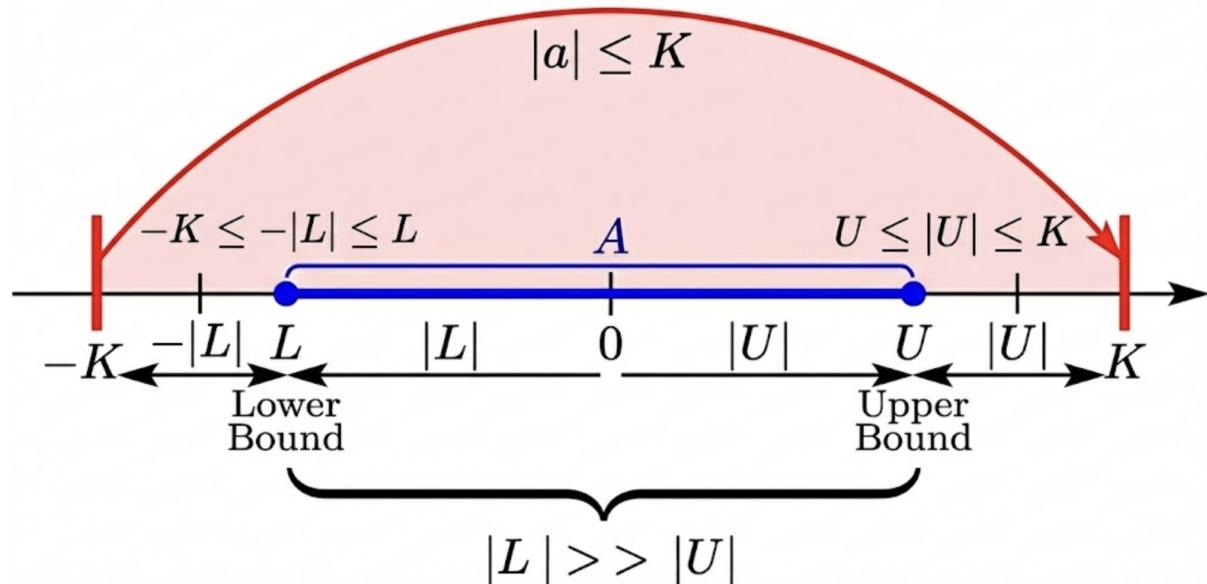
$$\forall a \in A, L \leq a \leq U$$

To show that A is bounded, we must find a single constant $k > 0$ such that $|a| \leq k$ for all $a \in A$.

Let $k = \max(|L|, |U|)$. Then:

- Since $k \geq |L|$, it follows that $-k \leq -|L|$. Because $-|L| \leq L$, we have $-k \leq L$.
- Since $k \geq |U|$ and $U \leq |U|$, we have $U \leq k$.

$$K = \max(|L|, |U|)$$



Therefore, we can write:

$$-k \leq L \leq a \leq U \leq k \Rightarrow -k \leq a \leq k$$

By the properties of absolute value, this implies $|a| \leq k$ for all $a \in A$. Thus, by definition, A is bounded.

■

2.2 E1.6 (b): Epsilon Test for Infimums

Set: $A = \left\{ \frac{\sqrt{x-1}}{x} \mid x \geq 2 \right\}$

1. *Supremum* ($\sup A$)

- **Value:** $\sup A = 1/2$
- **Finding the bound:** The maximum occurs when $x = 2$. Plugging this in: $\frac{\sqrt{2-1}}{2} = \frac{1}{2}$.
- **Epsilon Proof:** For any given $\varepsilon > 0$, we can choose the number $1/2$ from the set (where $x = 2$). Since ε is positive, it is always true that:

$$1/2 > 1/2 - \varepsilon$$

2. *Infimum* ($\inf A$)

- **Value:** $\inf A = 0$
- **Finding the bound:** The value of the expression approaches 0 as $x \rightarrow \infty$.
- **Epsilon Proof:** For any given $\varepsilon > 0$, we need to find a number in the set such that:

$$\frac{\sqrt{x-1}}{x} < 0 + \varepsilon$$

To simplify the search for x , we note that $\frac{\sqrt{x}}{x} > \frac{\sqrt{x-1}}{x}$. Therefore, any x that satisfies $\frac{\sqrt{x}}{x} < \varepsilon$ will also satisfy our condition.

Simplifying the expression:

$$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

Solving for x :

$$\frac{1}{\sqrt{x}} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < \sqrt{x} \Rightarrow \frac{1}{\varepsilon^2} < x$$

Choice of x : Pick $x = \frac{1}{\varepsilon^2} + 2$.

This value is valid because it is ≥ 2 (staying in the set) and is strictly $> \frac{1}{\varepsilon^2}$. This ensures the resulting value in the set is smaller than $0 + \varepsilon$.

